MADE EASY

Computer Science Engineering / IT

Toppers Handwritten Notes

Discrete Mathematics

By-Ramesh sundaram Sir

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DISCRETE MATHEMATICS

Kenneth Rosen

9 Marks

Chapters:

1. Logic
2. Combinatorics
3. Set Theory [KOLMAN, BUSAN $ ROSS]
4. Graph Theory [NARSINGH DEO] [Theory]

LOGIC

1. Logical StateMent ? [Proposition]
→ 2. Logical Operators & their properties ($\vee$, $\wedge$, $\neg$, $\Rightarrow$, $\Leftrightarrow$, $\oplus$ $\uparrow$ $\downarrow$)
   Disjunction, Conj., Negation
3. Tautology (T), Contradiction (C) & Contingency (CT)
   [Satisfiable / UnSatisfiable]
   (T or CT) (C)
→ 4. Normal Forms & properties: PDNF (Principle Disjunctive Normal Form)
   PCNF (Principle Conjunction Normal Form]
→ 5. Implications $ Biconditional ($\Rightarrow$, $\Leftrightarrow$)
6. Arguments & Fallacy [Invalid ArguMent]
7. Rules of Inference
→ 8. Predicate Logic - Quantifier ($\forall$, $\exists$)
   - Validity of a predicate
   - Properties
   - Translation

LOGIC

(S, ∘)
↓ ↳ operators
Set of all Logical Stmt

Logical StateMent - (Preposition)

- Declarative sentence which can be either true or false but not both.

Ex - This board is white.
This Fan is Rotating.
• This sentence is true.

[is/will tends to declaration]

Not a Logical StateMent

1. Questions - What is Your Name?
2. Command - Stand up.
3. Exclamation - Oh! That's great.
4. $x+2=4$
(it is not preposition bcoz for some x value it is true, it is false)
5. He is tall. (unless he is specified)
6. Today is Wednesday.
[Not a preposition bcoz today may be true, but tommorow it will become false]
7. Tommorow it will rain.
[Not a preposition.]
8. This sentence is false.
[Negative Self Referential Sentence]

Logical Operators:

A preposition is written in the following way:

$p: 2+2=4$

$q(x): x+2=4$ (Predicate) but not a preposition

~~True~~ False — $\forall x\, P(x)$] prepositions
True — $\exists x\, P(x)$

Negation - ($\sim, \bar{p}, \neg, p'$) . Unary operators

P	$\bar{P}$
0	1
1	0

P	Negation
is	is not
is not	is
$=$	$\neq$
$<$	$\geq$
$>$	$\leq$
$p \vee q$	$p' \wedge q' \Rightarrow p \downarrow q$
$p \wedge q$	$p' \vee q' \Rightarrow p \uparrow q$

P	Negation
$p \Rightarrow q$	pq'
$p \Leftrightarrow q$	$p \oplus q$
$p \oplus q$	$p \Leftrightarrow q$
$p \uparrow q$	$p \wedge q$
$p \downarrow q$	$p \vee q$

- if $p \vee q = 1$
  than $p = \sim q$ is one possibility but not the sure thing. it also allow some other thing
- if $p \wedge q = 0$
  $\Rightarrow [p = \sim q]$ not allways.
- If $p \vee q = 1$ & $p \wedge q = 0 \Rightarrow \boxed{p = \sim q}$
- If $p \Rightarrow 2+2=4$ or $3+7=10$
  $\bar{p} \Rightarrow 2+2 \neq 4$ and $3+7 \neq 10$
- If $p \Rightarrow 2+2=4$ and $3+7=10$
  $\bar{p} \Rightarrow 2+2 \neq 4$ OR $3+7 \neq 10$

$\wedge$ - all must be true
$\vee$ - atleast one is true
$\oplus$ - Exactly one is true
$\uparrow$ - Atleast one is false

- p : 2 is even & divisible by 4.
  p' : 2 is odd or not divisible by 4.
- p : if it rains, i will carry umbrella. [Either it does not rain OR I will carry Umbrella]
  p' : It rains and I will not carry Umbrella

Conversion of Secondary operators into Basic operators:

- $p \rightarrow q = p' + q$
- $p \Leftrightarrow q = p'q' + pq = (p \oplus q)' = p' \Leftrightarrow q' = p' \oplus q = p \oplus q'$
- $p \oplus q = pq' + qp' = p' \Leftrightarrow q = p \Leftrightarrow q' = p' \oplus q'$
- $p \Leftrightarrow q = (p' + q)(p + q')$ $[(p \Rightarrow q) \wedge (q \Rightarrow p)]$
- $p \oplus q = p \oplus q'$
- $p' \oplus q = p \Leftrightarrow q = p \oplus q'$

- p: A number is even if and only if divisible by 2. $[p \Rightarrow \bar{q} \wedge p \Leftarrow q]$
  P' : A number is even or it is divisible by 2, but not both.

- NOR - Neither ... NOR
- OR - Either ..... OR

Negation for predicate-

$P(x)$	$\sim P(x)$
$\forall x P(x)$	$\exists x \sim P(x)$
$\exists x P(x)$	$\forall x \sim P(x)$
$\forall x \sim P(x)$	$\exists x P(x)$
$\exists x (\sim P(x))$	$\forall x P(x)$

$\sim(\forall x(P(x) \to Q(x))) \equiv \exists x(\sim[P(x) \to Q(x)])$

$\equiv \exists x(P(x) \wedge \sim Q(x))$

$\sim(\forall x \exists y\, P(x,y)) \equiv \exists x \forall y \sim P(x,y)$

$\sim(\exists x \forall y \forall z\,(P(x,y,z) \oplus Q(x,y,z))) \equiv \forall x \exists y \exists y\,(P(x,y,z) \Leftrightarrow Q(x,y,z))$

$\sim(p \Rightarrow q) = \sim(p'+q)$

$p \Rightarrow q$	[stmt]	$(p=0 \text{ or } q=0) \Rightarrow (pq=0)$
$q \Rightarrow p$	[converse]	$(pq \neq 0) \Rightarrow (p=0 \text{ or } q=0)$
$\sim p \Rightarrow \sim q$	[inverse]	$(p \neq 0 \text{ and } q \neq 0) \Rightarrow (pq \neq 0)$
$\sim q \Rightarrow \sim p$	[contrapositive]	$(pq \neq 0) \Rightarrow (p \neq 0 \text{ and } q \neq 0)$

inary operators-

p	q	p+q $p \vee q$	p·q $p \wedge q$	$p \Rightarrow q$	$p \Leftrightarrow q$	$p \oplus q$	$p \uparrow q$	$p \downarrow q$
0	0	0	0	1	1	0	1	1
0	1	1	0	1	0	1	1	0
1	0	1	0	0	0	1	1	0
1	1	1	1	1	1	0	0	0

if two prepositions are equivalent (X, Y)

then $[X \Leftrightarrow Y \equiv 1]$ $[X \equiv Y \text{ iff } X \Leftrightarrow Y = 1]$

Q. Let $b \Leftrightarrow c$ and $a \Leftrightarrow (b \vee \neg b)$ is tautology
what can be inferred about $a \vee (b \wedge c)$?

$b \Leftrightarrow c \Rightarrow b \equiv c$

$a \Leftrightarrow (b \vee \sim b) \Rightarrow a = 1$

$\therefore a \vee (b \wedge c) = a \vee (b \wedge b) = 1 \vee b = 1$ (Tautology)

olean Algebra: $(S, +, \cdot, ')$
$(S, \vee, \wedge, \sim)$
$(S, \cup, \cap, A^c)$

[logic, Digital Logic, Set theory]

- No of elements in set of Boolean Algebra must be in power of 2.
- D_n is a boolean Algebra.

- $a - b = a \wedge b'$
- $A - (B \cup C) = (A-B) \cup (A-C)$ (Test T or F)

$a - (b+c) = (a-b) + (a-c)$

$ab'c' = ab' + ac'$ (false)

Properties of Operators- • operators are also known as logical connectives.

1. Closure- $\forall x, y, \begin{bmatrix} x+y \in S \\ x \cdot y \in S \\ \neg x \in S \end{bmatrix}$ OR $\begin{bmatrix} x \vee y \in S \\ x \wedge y \in S \\ \neg p \in S \end{bmatrix}$

$\forall A, B \in S \begin{bmatrix} A \cup B \in S \\ A \cap B \in S \\ A^c \in S \end{bmatrix}$

2. Commutative:

$\forall x, y \in S \begin{bmatrix} x+y = y+x \\ x \cdot y = y \cdot x \end{bmatrix}$ $\forall A, B \in S \begin{bmatrix} (A \cup B) = (B \cup A) \\ (B \cap A) = (A \cap B) \end{bmatrix}$

$\forall x, y \in S \begin{bmatrix} x \wedge y = y \wedge x \\ x \vee y = y \vee x \end{bmatrix}$

3. Associative:

$\forall x, y, z \in S \begin{bmatrix} x+(y+z) = (x+y)+z \\ x \cdot (y \cdot z) = (x \cdot y) \cdot z \end{bmatrix}$

$\forall x, y, z \in S \begin{bmatrix} x \wedge (y \wedge z) = (x \wedge y) \wedge z \\ x \vee (y \vee z) = (x \vee y) \vee z \end{bmatrix}$

$\forall A, B, C \in S \begin{bmatrix} A \cup (B \cup C) = (A \cup B) \cup C \\ (A \cap B) \cap C = A \cap (B \cap C) \end{bmatrix}$

4. Distributive: $\forall x, y, z \in S \begin{bmatrix} x+(y \cdot z) = (x+y)(x+z) \\ x \cdot (y+z) = xy + xz \end{bmatrix}$

$\forall x, y, z \in S \begin{bmatrix} x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \\ x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \end{bmatrix}$

$\forall \cancel{x, y, z}\ A, B, C \in S \begin{bmatrix} A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \end{bmatrix}$

5. Identity: $\begin{bmatrix} \exists_0 \forall x \quad x+0 = x = 0+x \\ \exists_1 \forall x \quad x \cdot 1 = x = 1 \cdot x \end{bmatrix}$ & $0 \neq 1$

$\exists \forall x \in S \begin{bmatrix} x \wedge T = x = T \wedge x \\ x \vee F = x = F \vee x \end{bmatrix}$

$\exists_\phi \forall A \in S \begin{bmatrix} A \cup \phi = A = \phi \cup A \\ A \cap S = A = S \cap A \end{bmatrix}$
$\exists_S \forall A \in S$

[S → Universal set]

Ques- The Smallest Finite Boolean Algebra has 2^n. what is value of n?

$n = 1 \Rightarrow 2^n \; \{0, 1\}$ — 0 ↳ Least Element, 1 → Greatest Elmt

$\left[\begin{array}{l} + \rightarrow \text{Least Upper Bound} \\ \cdot \rightarrow \text{Greatest Lower Bound} \end{array}\right]$

Complement-

$\forall x \, \exists x' \left[x + \bar{x} = 1 \right]$

$\forall x \, \exists x' \left[x \cdot x' = 0 \right]$

$\left[\begin{array}{l} x \wedge (\sim x) = F \\ x \vee (\sim x) = T \end{array}\right] \forall x \, \exists \sim x$

$\forall A \in S \; \exists A^c \in S \left[\begin{array}{l} A \cup A^c = S \\ A \cap A^c = \phi \end{array}\right]$

$\boxed{S\text{- Universal set}}$

Precedence of operators:

$[(\,) > (\,') > \wedge > \vee] \rightarrow$ for boolean Algebra

$\therefore (\,) > \,' > \cdot > + > \Rightarrow > \Leftrightarrow$

Ex- $\left(\left[\left(p \vee \left((\sim q) \wedge r\right)\right) \Rightarrow s\right] \Leftrightarrow t\right)$

If an element satisfy complement property, it surely satisfies identity property.

Laws for Boolean Algebra-

Idempotent Law- $\forall p \in S \left[\begin{array}{l} p + p = p \\ p \cdot p = p \end{array}\right]$ $\forall p \in S \left[\begin{array}{l} p \wedge p = p \\ p \vee p = p \end{array}\right]$

$\forall A \in S \left[\begin{array}{l} A \cup A = A \\ A \cap A = A \end{array}\right]$

- The biggest polynomial in Boolean Algebra with degree 1.
- No power, No coefficient exist

Absorbtion Law

$\forall p \in S \left[\begin{array}{l} p + pq = p \\ p \cdot (p+q) = p \end{array}\right]$ $\left[\begin{array}{l} p + qp = p = p + pq \\ p \cdot (p+q) = p = (p+q) \cdot p \end{array}\right]$

$$\left[\begin{array}{l} \text{Ex - } \underbrace{pq + pq}r + prst + p \\ = pq + pqst + p \\ = pq + p = p \end{array}\right]$$

$$\text{Ex4} \left[\begin{array}{l} p + [(p+q')(q'+s')] \\ \neq p \end{array}\right]$$

$$\text{Ex} \left[\begin{array}{l} p \cdot (q + p'r) \\ \neq p \end{array}\right]$$

$$\text{Ex3} \left[p \cdot (q + r's' + p + t' + q'r') = p \right]$$

$\forall p \in S \quad \begin{bmatrix} p + p'q = p+q \\ p(p'+q) = pq \end{bmatrix} \qquad \forall p \in S \begin{bmatrix} p' + pq = p'+q \\ p'(p+q) = p'q \end{bmatrix}$

Ex1- $p + q'r + r + p'q$
$= r + p + q$

3) <u>DeMorgan's Law</u>: $\forall p,q \in S \begin{bmatrix} (p+q)' = p'q' \\ (p \cdot q)' = p'+q' \end{bmatrix} \quad \begin{bmatrix} \sim(p \vee q) = \sim p \wedge \sim q \\ \sim(p \wedge q) = \sim p \vee \sim q \end{bmatrix}$

Ex- $(p+qr) \Rightarrow s$
Simplified form is $[\,[p+qr]'+s = p' \cdot (q'+r') + s\,]$

<u>for Set Theory</u>- $\forall A,B \in S \begin{bmatrix} (A \cup B)^c = A^c \cap B^c \\ (A \cap B)^c = A^c \cup B^c \end{bmatrix}$

4. <u>Law of Double Complement</u>:

$\forall p \in S \quad (p')' = p$

$\forall A \in S \quad (A^c)^c = A$

- $p' = q$ if and only if $q' = p$
  $[p' = q \Leftrightarrow q' = p]$. Always a tautology

5. <u>Domination Law</u>- $\forall x \in S \begin{bmatrix} x+1 = 1 \\ x \cdot 0 = 0 \end{bmatrix}$

$\forall p \in S \begin{bmatrix} x \vee 1 = 1 \\ x \wedge 0 = 0 \end{bmatrix}$

$\forall A \in S \begin{bmatrix} A \cup U = U \\ A \cap \phi = \phi \end{bmatrix}$ U- universal set

<u>Note</u>- 1. $\underbrace{[p' \Rightarrow q] \equiv [q' \Rightarrow p]}_{\text{contrapositive set}} \equiv (p \vee q)$

Tautology, Contradiction & Consistency

Ques- Which of the following is tautology?

(a) ~~$p \Rightarrow q$~~ $p \vee q \Rightarrow p \wedge q$ — (a) $(p+q) \Rightarrow pq = p'q' + pq = p \Leftrightarrow q$

(b) $p \vee q \Rightarrow r \wedge p$ — (b) $(p+q) \Rightarrow rp = p'q' + pr$

(c) $p \vee q \Rightarrow r \vee s$ — (c) $(p+q) \Rightarrow r+s = p'q' + r + s$

(d) $s \Rightarrow s \wedge t$ — (d) $s' + st = s' + t$

✓(e) None

Tautology- for a preposition, if truth table contain all true values only

OR

$[p \Leftrightarrow 1 \equiv 1]$ • Also known as valid preposition.

Ques- Check whether it is a tautology?

$$\left[\left[(p \Rightarrow q) \wedge (q \Rightarrow r)\right] \Rightarrow (r \Rightarrow p)\right]$$

$$\left[\left[(p'+q)\,(q'+r)\right] \Rightarrow (r'+p)\right]$$

$$= (p'+q)' + (q'+r)' + r' + p$$

$$= pq' + qr' + r' + p$$

$$= p + r' \quad \Rightarrow \text{Hence it is contigency}$$

Ques- Check whether it is tautology?

$$\left[((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)\right]$$

$$= pq' + qr' + p' + r$$

$$= p' + q' + r + q = 1 \quad \text{Hence it is a tautology}$$

contradiction- for all value, it is false $[p \Leftrightarrow 0 \equiv 0]$

• Also known as fallacy.

• Also known as invalid.

[Preposition]

- [Satisfiable (SAT)]
  - [Tautology] (T) [All true]
  - [contingency] (CT) $\neq 0$ or 1
- [UnSatisfiable (USAT)]
  - [Contradiction] (C) $= 0$